

Integrable Discretization of Soliton Equations via Bilinear Method and Bäcklund Transformation

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Abstract In this paper, we present a systematic procedure to derive discrete analogues of integrable PDEs via Hirota's bilinear method. This approach is mainly based on the compatibility between an integrable system and its Bäcklund transformation. We apply this procedure to several equations, including the extended Korteweg-de-Vries (KdV) equation, the extended Kadomtsev-Petviashvili (KP) equation, the extended Boussinesq equation, the extended Sawada-Kotera (SK) equation and the extended Ito equation, and obtain their associated semi-discrete analogues. In the continuum limit, these differential-difference systems converge to their corresponding smooth equations. For these new integrable systems, their Bäcklund transformations and Lax pairs are derived.

Keywords: Integrable discretization, bilinear method, Bäcklund transformation

1 Introduction

The integrable discretization of soliton equations or integrable systems has attracted a lot of interest [11, 25] and has been studied for many years from various viewpoints. Although there is no commonly accepted unified mathematical definition of integrable systems, the most widely accepted features of these kinds of equations include the zero curvature representation or Lax pair, Bäcklund transformation and multi-soliton solutions.

The problem of integrable discretization is to discretize an integrable differential equation, meanwhile preserving its integrability. Ablowitz and Ladik [1], Hirota [14–16], Nijhoff [30, 35] and others did pioneer work on the integrability of difference and differential-difference equations over 30 years ago. As stated by Kakei Saburo in [24], besides the bilinear methods [3–7, 14–16], there are several other famous procedures for integrable discretization, including

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- discretization of Lax pairs [1]
- discretization based on the dressing method [26]
- the direct linearization method [30, 35]
- discretization based on the construction of Poisson structure [8, 37].

Here we have only given a few representative references. In addition, there is some recent progress in [36] where a method based on loop group is introduced. Other recent works can be found in [9, 10, 21, 32–34, 38–42] and references there.

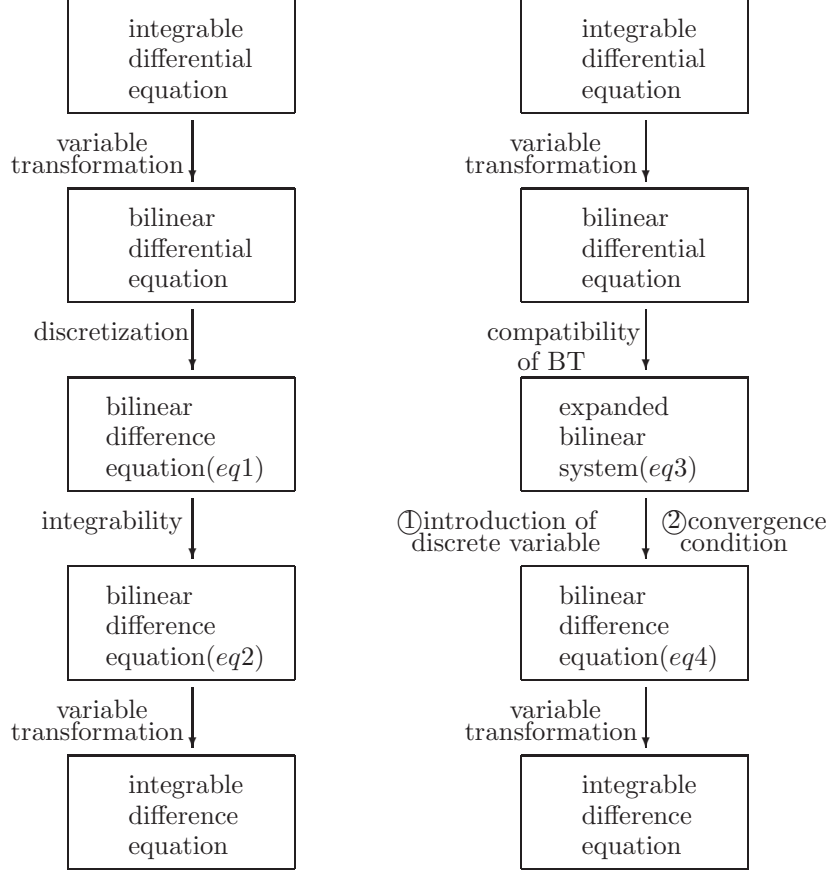


Fig. 1: Flow charts of bilinear method of integrable discretization – Left chart: traditional approach; Right chart: new approach

There is a well-known idea that integrable discretizations are provided by a suitable interpretation of Bäcklund transformations. Generating integrable differential-difference equations from Bäcklund transformations, as we know, was first studied by Chiu and Ladik in [2] and Levi and Benguria in [27, 28]. Here in this paper, we showed that, by introducing a convergence condition, the integrable discretization of a soliton equation can be derived from the compatibility between the integrable differential equation and its Bäcklund transformation.

The procedure presented here is from a different viewpoint of the bilinear method and is much more direct. The traditional bilinear method [14, 18, 22, 31]

as shown in the left chart of Fig. 1, is to discretize the bilinear form of the soliton equation first and then confirm the integrability. Due to non-uniqueness, there is some freedom in (eq1) when we discretize the smooth bilinear differential equation. Subsequently (eq2) is obtained from (eq1) by considering its integrabilities (soliton solutions or Bäcklund transformation). Different from the traditional approach, the new procedure shown on the right of Fig. 1 preserves the integrability first and then discretize the equation. Given an integrable bilinear equation and its Bäcklund transformation, an expanded system (eq3) of bilinear equations compatible with the original bilinear equation can be obtained. In (eq3), there are some free parameters inherited from the associated Bäcklund transformation. Using properties of the τ -function or the ideas in [27], it is natural to introduce some discrete variables in the new integrable system. By considering continuum limit, we impose a convergence condition on the new integrable system. Then the parameters can be determined and an integrable discretization eq4 of the original bilinear system can be derived. We will illustrate this approach with the extended Korteweg-de-Vries (KdV) system.

The extended KdV system is given by

$$v_t = \frac{1}{4}q + \frac{3}{2}u^2, \quad (1.1)$$

$$u_t = \frac{1}{4}r + 3up, \quad (1.2)$$

$$p = u_x, q = p_x, r = q_x. \quad (1.3)$$

The dependent variable transformation $v = (\ln f)_x$, $u = (\ln f)_{xx}$ produce the bilinear form

$$D_x(D_t - \frac{1}{4}D_x^3)f \cdot f = 0. \quad (1.4)$$

Here the D -operator is defined by

$$D_t^m D_x^n a(t, x) \cdot b(t, x) = \frac{\partial^m}{\partial s^m} \frac{\partial^n}{\partial y^n} a(t + s, x + y) b(t - s, x - y) \Big|_{s=0, y=0},$$

$$m, n = 0, 1, 2, \dots, \quad (1.5)$$

or by the exponential identity

$$\begin{aligned} \exp(\delta D_z) a(z) \cdot b(z) &= \exp(\delta \partial_y) (a(z + y) b(z - y)) \Big|_{y=0}, \\ &= a(z + \delta) b(z - \delta). \end{aligned} \quad (1.6)$$

In the following, we will call the extended KdV system (1.1)-(1.3) and its bilinear form (1.4) KdV equation without diffusion. The following bilinear Bäcklund transformation of (1.4) has been given in [13]:

$$(D_x^2 - \lambda D_x) f \cdot g = 0, \quad (1.7)$$

$$(D_t - \frac{1}{4}D_x^3 + \mu) f \cdot g = 0, \quad (1.8)$$

where λ, μ are arbitrary constants. For details of the D -operator and the bilinear Bäcklund transformation, see [13]. Here we consider (1.4) and (1.7) together as a new system

$$D_x(D_t - \frac{1}{4}D_x^3)f \cdot f = 0, \quad (1.9)$$

$$(D_x^2 - \lambda D_x)f \cdot g = 0. \quad (1.10)$$

Taking $f \rightarrow f_n$ and $g \rightarrow f_{n-h}$, where n is a discrete variable and h is the step size, we get a differential-difference system

$$D_x(D_t - \frac{1}{4}D_x^3)f_n \cdot f_n = 0, \quad (1.11)$$

$$(D_x^2 - \lambda D_x)f_n \cdot f_{n-h} = 0. \quad (1.12)$$

From properties of the Bäcklund transformation, the above system is also integrable.

Remark 1 *It will be better to write f_{n+h} as f_{n+1} . For the sake of introducing a convergence condition $D_x = D_n + O(h)$, we keep using the form f_{n+h} .*

To obtain the integrable discretization of (1.4), the convergence condition $D_x = D_n + O(h^k)$, $k \geq 1$ should be upheld. One can rewrite (1.12) as

$$[D_x^2 \cosh(\frac{h}{2}D_n) - \lambda D_x \sinh(\frac{h}{2}D_n)]f_n \cdot f_n = 0. \quad (1.13)$$

Expanding this equation in powers of h , we get

$$[D_x^2 - \frac{h}{2}\lambda D_x D_n + O(h^2)]f_n \cdot f_n = 0. \quad (1.14)$$

By imposing the convergence condition $D_x = D_n + O(h^k)$, $k \geq 1$, we have $\lambda = \frac{2}{h}$.

Remark 2 *With the convergence condition $D_x = D_n + O(h^k)$, $k \geq 1$, the discrete variable n can be viewed as an approximation to x and h is just the step size in the x -direction. We do not discretize x directly but take it as an auxiliary variable. In the bilinear equations, we still write it as x without confusion.*

We next show that

$$D_x(D_t - \frac{1}{4}D_x^3)f_n \cdot f_n = 0, \quad (1.15)$$

$$(D_x^2 - \frac{2}{h}D_x)f_n \cdot f_{n-h} = 0, \quad (1.16)$$

is an integrable discretization of the KdV equation. Applying the dependent variable transformation $v_n = (\ln f_n)_x$, $u_n = v_{n,x}$, $p_n = u_{n,x}$, $q_n = p_{n,x}$, $r_n = q_{n,x}$

to (1.15)-(1.16), we get

$$u_{n,t} = \frac{1}{4}r_n + 3u_n p_n, \quad (1.17)$$

$$v_{n,t} = \frac{1}{4}q_n + \frac{3}{2}u_n^2, \quad (1.18)$$

$$\begin{aligned} (p_{n+\frac{h}{2}} + p_{n-\frac{h}{2}}) &= \frac{2}{h}(u_{n+\frac{h}{2}} - u_{n-\frac{h}{2}}) - 2(u_{n+\frac{h}{2}} - u_{n-\frac{h}{2}})(v_{n+\frac{h}{2}} - v_{n-\frac{h}{2}}) \\ (q_{n+\frac{h}{2}} + q_{n-\frac{h}{2}}) &= \frac{2}{h}(p_{n+\frac{h}{2}} - p_{n-\frac{h}{2}}) - 2(p_{n+\frac{h}{2}} - p_{n-\frac{h}{2}})(v_{n+\frac{h}{2}} - v_{n-\frac{h}{2}}) \\ &\quad - 2(u_{n+\frac{h}{2}} - u_{n-\frac{h}{2}})^2, \end{aligned} \quad (1.20)$$

$$\begin{aligned} (r_{n+\frac{h}{2}} + r_{n-\frac{h}{2}}) &= \frac{2}{h}(q_{n+\frac{h}{2}} - q_{n-\frac{h}{2}}) - 6(p_{n+\frac{h}{2}} - p_{n-\frac{h}{2}})(u_{n+\frac{h}{2}} - u_{n-\frac{h}{2}}) \\ &\quad - 2(q_{n+\frac{h}{2}} - q_{n-\frac{h}{2}})(v_{n+\frac{h}{2}} - v_{n-\frac{h}{2}}). \end{aligned} \quad (1.21)$$

In the above relations among v_n, u_n, p_n, q_n, r_n , equations (1.17)-(1.18) are derived from (1.15), whereas equations (1.19)-(1.21) come from (1.16). Replacing u_n by $u(x, t)$, u_{n+h} by $u(x+h, t)$, and similarly for the other variables, if we take the limit $h \rightarrow 0$ in the above equations, we get $u_t = \frac{1}{4}r + 3up$, $v_t = \frac{1}{4}q + 3u^2/2$, $p = u_x$, $q = p_x$, $r = q_x$. Thus (1.17)-(1.21) construct a discretization of the KdV equation (1.1)-(1.3). From the discussion above, we have the following theorem:

Theorem 1.1 *The system (1.15)-(1.16) is an integrable discretization of the KdV equation (1.4). Using transformations $v_n = (\ln f_n)_x$, $u_n = v_{n,x}$, $p_n = u_{n,x}$, $q_n = p_{n,x}$, this system can be converted to (1.17)-(1.21), which converge to the KdV equation (1.1)-(1.3) as $h \rightarrow 0$.*

As an integrable system, we derived its bilinear Bäcklund transformation.

Proposition 1 *The bilinear equations (1.15) and (1.16) have the Bäcklund transformation (BT)*

$$(D_x e^{-\frac{h}{2}D_n})f_n \cdot g_n = (-\frac{1}{h}e^{-\frac{h}{2}D_n} + \beta e^{\frac{h}{2}D_n})f_n \cdot g_n, \quad (1.22)$$

$$D_x^2 f_n \cdot g_n = \gamma f_n g_n, \quad (1.23)$$

$$(D_t - \frac{1}{4}D_x^3 - \frac{3\gamma}{4}D_x)f_n \cdot g_n = 0, \quad (1.24)$$

where β and γ are arbitrary constants.

Proof. Let f_n be a solution of equations (1.15)-(1.16). If we can show that the g_n given by (1.22)-(1.24) satisfies

$$P_1 \equiv D_x(D_t - \frac{1}{4}D_x^3)g_n \cdot g_n = 0,$$

$$P_2 \equiv (D_x^2 - \frac{2}{h}D_x)e^{\frac{h}{2}D_n}g_n \cdot g_n = 0,$$

then (1.22)-(1.24) form a BT for (1.15)-(1.16). It has been shown in [13] that equations (1.23)-(1.24) construct a BT for $D_x(D_t - \frac{1}{4}D_x^3)f \cdot f = 0$. Thus $P_1 = 0$ is obvious. As for P_2 , by using (1.22)-(1.24) and the bilinear identities (A.1)-(A.3), we can precisely calculate that

$$\begin{aligned}
& -[e^{\frac{D_n}{2}} f_n \cdot f_n] P_2 \\
& \equiv [(D_x^2 - \frac{2}{h} D_x) e^{\frac{h}{2} D_n} f_n \cdot f_n] [e^{\frac{D_n}{2}} g_n \cdot g_n] \\
& \quad - [e^{\frac{D_n}{2}} f_n \cdot f_n] [(D_x^2 - \frac{2}{h} D_x) e^{\frac{h}{2} D_n} g_n \cdot g_n] \\
& = D_x [(D_x e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (e^{-\frac{h}{2} D_n} f_n \cdot g_n) - (e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (D_x^{-\frac{h}{2} D_n} f_n \cdot g_n)] \\
& \quad - \frac{2}{h} D_x (e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (e^{-\frac{h}{2} D_n} f_n \cdot g_n) \\
& = D_x [(D_x e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (e^{-\frac{h}{2} D_n} f_n \cdot g_n) + (e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (D_x^{-\frac{h}{2} D_n} f_n \cdot g_n)] \\
& \quad - 2 D_x (e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (D_x e^{-\frac{h}{2} D_n} f_n \cdot g_n + \frac{1}{h} e^{-\frac{h}{2} D_n} f_n \cdot g_n) \\
& = 2 \sinh(\frac{h}{2} D_n) [(D_x^2) f_n \cdot g_n] \cdot (f_n g_n) = 0.
\end{aligned}$$

Thus we have completed the proof of **Proposition 1**.

Remark 3 *The way to derive Bäcklund transformations we used here is constructed by Hirota in [12].*

Setting $v_n = (\ln g_n)_x$, $u_n = v_{n,x}$, $p_n = u_{n,x}$, $q_n = p_{n,x}$, $f_n = \phi_n g_n$ and $\psi_n = \phi_{n,x}$ in (1.22)-(1.24), we can get a Lax pair for the discrete equations (1.17)-(1.21):

$$\begin{aligned}
\beta \begin{pmatrix} \phi_{n+1} \\ \psi_{n+1} \end{pmatrix} &= \begin{pmatrix} \frac{1}{h} + v_n - v_{n+1} & 1 \\ \gamma - u_n - u_{n+1} & \frac{1}{h} + v_n - v_{n+1} \end{pmatrix} \begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix}, \\
\begin{pmatrix} \phi_{n,t} \\ \psi_{n,t} \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2} p_n & \gamma + u_n \\ -\frac{1}{2} q_n + (\gamma - 2u_n)(\gamma + u_n) & \frac{1}{2} p_n \end{pmatrix} \begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix}.
\end{aligned}$$

Equations (1.17)-(1.21) can be obtained from the compatibility of the two linear problems above.

We summarize the procedure in the following algorithm:

Algorithm 1 *Given an integrable equation, let x be the smooth variable, n be the corresponding discrete variable and h be the step size.*

Step 1. *Transform the integrable system to bilinear form via variable transformations.*

Step 2. *Derive the bilinear Bäcklund transformation and construct a new compatible system from the original equation and the Bäcklund transformation.*

Step 3. *Choose parameters for the Bäcklund transformation to satisfy the convergence condition*

$$D_x = D_n + O(h^k), \quad k \geq 1.$$

Step 4. *Transform the new integrable system to nonlinear form and check whether the nonlinear form converges to the original equation as $h \rightarrow 0$.*

Remark 4 *In the case of the KdV equation, only the first equation (1.7) of the Bäcklund transformation is used to construct a new compatible system. However, in other cases like the Kadomtsev-Petviashvili equation (see Section 2), all member equations of its Bäcklund transformation may be needed to make the discrete system closed and one may even employ other compatible equations to construct the integrable discretization.*

We apply the newly proposed procedure to several equations. In the following 4 sections, we consider the extended Kadomtsev-Petviashvili (KP) equation, the extended Boussinesq equation, the extended Sawada-Kotera equation and the extended Ito equation, respectively.

2 Kadomtsev-Petviashvili Equation

In this section, we attempt to use the new approach to derive a discretization of the extended KP equation

$$4u_t - r - 12up - 3v_{yy} = 0, \quad (2.1)$$

$$p = u_x, q = p_x, r = q_x, u_y = v_{xy}. \quad (2.2)$$

The dependent variable transformation

$$u = (\ln f)_{xx} \quad (2.3)$$

gives the bilinear form

$$(D_x^4 - 4D_x D_t + 3D_y^2) f \cdot f = 0. \quad (2.4)$$

We will call the system (2.1)-(2.2) and its bilinear form (2.4) KP equation when there is no ambiguity. A bilinear Bäcklund transformation of (2.4) is

$$(D_y - D_x^2 - \mu D_x) f \cdot g = 0, \quad (2.5)$$

$$(3D_y D_x - 4D_t + D_x^3 + 3\mu D_y) f \cdot g = 0, \quad (2.6)$$

where μ is an arbitrary constant [29].

Setting $f \rightarrow f_n$ and $g \rightarrow f_{n-h}$ in (2.4), (2.5) and (2.6), we get a compatible system

$$(D_x^4 - 4D_x D_t + 3D_y^2) f_n \cdot f_n = 0, \quad (2.7)$$

$$(D_y - D_x^2 - \mu D_x) f_n \cdot f_{n-h} = 0, \quad (2.8)$$

$$(3D_y D_x - 4D_t + D_x^3 + 3\mu D_y) f_n \cdot f_{n-h} = 0, \quad (2.9)$$

where h is the step size. Now rewrite (2.8) as

$$[D_y \sinh(\frac{h}{2} D_n) - D_x^2 \cosh(\frac{h}{2} D_n) - \mu D_x \sinh(\frac{h}{2} D_n)] f_n \cdot f_n = 0.$$

Expanding this equation in powers of h , we have

$$[D_x^2 + \frac{h}{2}\mu D_x D_n + O(h)]f_n \cdot f_n = 0. \quad (2.10)$$

By considering the convergence condition $D_x = D_n + O(h^k)$, $k \geq 1$, we have $\mu = -\frac{2}{h}$. Thus we get an integrable differential-difference system

$$(D_x^4 - 4D_x D_t + 3D_y^2)f_n \cdot f_n = 0, \quad (2.11)$$

$$(D_y - D_x^2 + \frac{2}{h}D_x)f_n \cdot f_{n-h} = 0, \quad (2.12)$$

$$(3D_y D_x - 4D_t + D_x^3 - \frac{6}{h}D_y)f_n \cdot f_{n-h} = 0. \quad (2.13)$$

We move on to show that this system (viewing x as an auxiliary variable) converges to the KP equation in nonlinear form as $h \rightarrow 0$. Letting $w_n = \ln(f_n)$, $v_n = w_{n,x}$, $u_n = v_{n,x}$, $p_n = u_{n,x}$, $q_n = p_{n,x}$, $r_n = q_{n,x}$, equations (2.11)-(2.13) can be transformed to

$$4u_{n,t} - r_n - 12u_n p_n - 3v_{n,yy} = 0, \quad (2.14)$$

$$u_{n+h} + u_n = \frac{2}{h}(v_{n+h} - v_n) + (w_{n+h,y} - w_{n,y}) - (v_{n+h} - v_n)^2, \quad (2.15)$$

$$p_{n+h} + p_n = \frac{2}{h}(u_{n+h} - u_n) + (v_{n+h,y} - v_{n,y}) - 2(v_{n+h} - v_n)(u_{n+h} - u_n), \quad (2.16)$$

$$q_{n+h} + q_n = \frac{2}{h}(p_{n+h} - p_n) + (u_{n+h,y} - u_{n,y}) - 2(v_{n+h} - v_n)(p_{n+h} - p_n) - 2(u_{n+h} - u_n)^2, \quad (2.17)$$

$$r_{n+h} + r_n = \frac{2}{h}(q_{n+h} - q_n) + (p_{n+h,y} - p_{n,y}) - 2(v_{n+h} - v_n)(q_{n+h} - q_n) - 6(u_{n+h} - u_n)(p_{n+h} - p_n), \quad (2.18)$$

$$\begin{aligned} & 3((v_{n+h,y} + v_{n,y}) + (w_{n+h,y} - w_{n,y})(v_{n+h} - v_n)) - 4(w_{n+h,t} - w_{n,t}) \\ & + (p_{n+h} - p_n) + 3(v_{n+h} - v_n)(u_{n+h} + u_n) + (v_{n+h} - v_n)^3 \\ & = \frac{6}{h}(w_{n+h,y} - w_{n,y}). \end{aligned} \quad (2.19)$$

Deriving an analog of (2.19) with x instead and eliminating w by using (2.15), we get

$$\begin{aligned} & 3(u_{n+h,y} + u_{n,y}) + 6(u_{n+h} + u_n)(u_{n+h} - u_n) + 6(u_{n+h} - u_n)(v_{n+h} - v_n)^2 \\ & + 3(v_{n+h,y} - v_{n,y})(v_{n+h} - v_n) - 4(v_{n+h,t} - v_{n,t}) + (q_{n+h} - q_n) \\ & + 3(v_{n+h} - v_n)(p_{n+h} + p_n) - \frac{6}{h}(u_{n+h} - u_n)(v_{n+h} - v_n) \\ & = \frac{6}{h}(v_{n+h,y} - v_{n,y}). \end{aligned} \quad (2.20)$$

Replacing v_n by $v(x, y, t)$, v_{n+h} by $v(x + h, y, t)$, and similarly for u , p , q , r and then taking the limit $h \rightarrow 0$, equations (2.14), (2.16)-(2.18) and (2.20) become

$$4u_t - r - 12up - 3v_{yy} = 0, \quad (2.21)$$

$$p = u_x, q = p_x, r = q_x, u_y = v_{xy}. \quad (2.22)$$

From the discussion above, we have the following theorem for the KP equation:

Theorem 2.1 *The system (2.11)-(2.13) is an integrable discretization of the KP equation (2.4). Using transformations $v_n = w_{n,x}$, $u_n = v_{n,x}$, $p_n = u_{n,x}$, $q_n = p_{n,x}$, $r_n = q_{n,x}$, this system can be converted to (2.14), (2.16)-(2.18) and (2.20), which converge to the KP equation (2.1)-(2.2) as $h \rightarrow 0$.*

Remark 5 *Unlike the KdV equation case, we need both equations of the Bäcklund transformation to construct the integrable discretization of the KP equation. Note that there are 5 variables v_n , u_n , p_n , q_n , r_n in (2.14), (2.16)-(2.18) and (2.20). The purpose of (2.20) is to make the system closed.*

We have the following proposition for (2.11)-(2.13):

Proposition 2 *Bilinear equations (2.11)-(2.13) have the Bäcklund transformation*

$$(D_x e^{-\frac{h}{2}D_n})f_n \cdot g_n = (-\frac{1}{h}e^{-\frac{h}{2}D_n} + \beta e^{\frac{h}{2}D_n})f_n \cdot g_n, \quad (2.23)$$

$$(D_y - D_x^2)f_n \cdot g_n = \gamma f_n g_n, \quad (2.24)$$

$$(3D_y D_x - 4D_t + D_x^3 - 3\gamma D_x)f_n \cdot g_n = 0, \quad (2.25)$$

where β and γ are arbitrary constants.

Proof. Let f_n be a solution of (2.11)-(2.12) and g_n be given by (2.23)-(2.25). In a way similar to the proof of Proposition 1, we have

$$P_1 \equiv (D_x^4 - 4D_x D_t + 3D_y^2)g_n \cdot g_n = 0,$$

$$P_2 \equiv (D_y - D_x^2 + \frac{2}{h}D_x)e^{\frac{h}{2}D_n}g_n \cdot g_n = 0.$$

Thus it suffices to show that

$$P_3 \equiv (3D_y D_x - 4D_t + D_x^3 - \frac{6}{h}D_y)e^{\frac{h}{2}D_n}g_n \cdot g_n = 0.$$

In fact, by using $P_2 = 0$, equations (2.23)-(2.25) and the bilinear identities (A.1)-(A.7), we can precisely deduce

$$\begin{aligned} & -[e^{\frac{D_n}{2}}f_n \cdot f_n]P_3 \\ & \equiv [(3D_y D_x - 4D_t + D_x^3 - \frac{6}{h}D_y)e^{\frac{h}{2}D_n}f_n \cdot f_n][e^{\frac{D_n}{2}}g_n \cdot g_n] \\ & \quad - [e^{\frac{D_n}{2}}f_n \cdot f_n][(3D_y D_x - 4D_t + D_x^3 - \frac{6}{h}D_y)e^{\frac{h}{2}D_n}g_n \cdot g_n] \\ & = 2 \sinh(\frac{h}{2}D_n)[(3D_x D_y - 4D_t + D_x^3 - 3\gamma D_x)f_n \cdot g_n] \cdot (f_n g_n) = 0. \end{aligned}$$

Hence the proof is complete.

Setting $v_n = (\ln g_n)_x$, $u_n = v_{n,x}$, $p_n = u_{n,x}$, $q_n = p_{n,x}$, $f_n = \phi_n g_n$ and $\psi_n = \phi_{n,x}$ in (2.23)-(2.25), we can get a Lax pair for the discrete equations (2.14), (2.16)-(2.18) and (2.20), namely,

$$\beta \begin{pmatrix} \phi_{n+1} \\ \psi_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{h} + v_n - v_{n+1} & 1 \\ \partial_y - u_n - u_{n+1} - \gamma & \frac{1}{h} + v_n - v_{n+1} \end{pmatrix} \begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix}$$

and

$$\begin{pmatrix} \phi_{n,t} \\ \psi_{n,t} \end{pmatrix} = \begin{pmatrix} \frac{3}{2}v_{n,y} - \frac{1}{2}p_n & \partial_y - \gamma + u_n \\ M_{21} & \frac{3}{2}v_{n,y} + \frac{1}{2}p_n \end{pmatrix} \begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix}.$$

where $M_{21} = -\frac{1}{2}u_{n,y} - \frac{1}{2}q_n + (\gamma - u_n)(\gamma + 2u_n) - (2\gamma + u_n)\partial_y + \partial_y^2$.

3 Boussinesq equation

The extended Boussinesq equation is

$$u_{tt} - q - s - 12uq - 12p^2 = 0, \quad (3.1)$$

$$v_{tt} - p - r - 12up = 0, \quad (3.2)$$

$$p = u_x, q = p_x, r = q_x, s = r_x. \quad (3.3)$$

Under the dependent variable transformation

$$u = (\ln f)_{xx}, v = (\ln f)_x, \quad (3.4)$$

equation (3.1)-(3.3) can be transformed to the bilinear form

$$(D_t^2 - D_x^2 - D_x^4)f \cdot f = 0, \quad (3.5)$$

We will still call the extended Boussinesq equation and its bilinear form Boussinesq equation without confusion. A two-parameter Bäcklund transformation has been given in [17]:

$$(D_t - aD_x^2 + \xi D_x)f \cdot g = 0, \quad (3.6)$$

$$(-aD_x D_t + D_x^3 + \xi a D_x^2 + (1 - \xi^2)D_x - \eta a)f \cdot g = 0, \quad (3.7)$$

where $a^2 = -3$ and ξ, η are arbitrary parameters. We can rewrite (3.7) as

$$(3D_x D_t + aD_x^3 + aD_x + a\xi D_t + 3\eta)f \cdot g = 0. \quad (3.8)$$

Setting $\eta = 0$, $\xi = \frac{2a}{h}$, $f \rightarrow f_n$, $g \rightarrow f_{n-h}$ in (3.5), (3.6), we get a compatible differential-difference system

$$(D_t^2 - D_x^2 - D_x^4)f_n \cdot f_n = 0, \quad (3.9)$$

$$(D_t - aD_x^2 + \frac{2a}{h}D_x)f_n \cdot f_{n-h} = 0. \quad (3.10)$$

Here $\eta = 0$, $\xi = \frac{2a}{h}$ are obtained from the convergence condition $D_x = D_n + O(h^k)$.

Viewing x as an auxiliary variable, substituting $w_n = 2\ln(f_n)$, $v_n = w_{n,x}$, $u_n = v_{n,x}$, $p_n = u_{n,x}$, $q_n = p_{n,x}$, $r_n = q_{n,x}$ into (3.9)-(3.10), we can get

$$u_{n,tt} - q_n - s_n - 12u_n q_n - 12p_n^2 = 0, \quad (3.11)$$

$$v_{n,tt} - p_n - r_n - 12u_n p_n = 0, \quad (3.12)$$

$$\begin{aligned} a(p_{n+h} + p_n) &= \frac{2a}{h}(u_{n+h} - u_n) + (v_{n+h,t} - v_{n,t}) \\ &\quad - 2a(u_{n+h} - u_n)(v_{n+h} - v_n), \end{aligned} \quad (3.13)$$

$$\begin{aligned} a(q_{n+h} + q_n) &= \frac{2a}{h}(p_{n+h} - p_n) + (u_{n+h,t} - u_{n,t}) \\ &\quad - 2a(p_{n+h} - p_n)(v_{n+h} - v_n) - 2a(u_{n+h} - u_n)^2, \end{aligned} \quad (3.14)$$

$$\begin{aligned} a(r_{n+h} + r_n) &= \frac{2a}{h}(q_{n+h} - q_n) + (p_{n+h,t} - p_{n,t}) \\ &\quad - 6a(p_{n+h} - p_n)(u_{n+h} - u_n) \\ &\quad - 2a(q_{n+h} - q_n)(v_{n+h} - v_n), \end{aligned} \quad (3.15)$$

$$\begin{aligned} a(s_{n+h} + s_n) &= \frac{2a}{h}(r_{n+h} - r_n) + (q_{n+h,t} - q_{n,t}) - 6a(p_{n+h} - p_n)^2 \\ &\quad - 8a(q_{n+h} - q_n)(u_{n+h} - u_n) \\ &\quad - 2a(r_{n+h} - r_n)(v_{n+h} - v_n). \end{aligned} \quad (3.16)$$

Here (3.13)-(3.16) are all derived from (3.10).

Replacing u_n by $u(x, t)$, u_{n+h} by $u(x+h, t)$, and similarly for w, v, p, q, r, s and then taking the limit $h \rightarrow 0$, equations (3.11)-(3.16) become

$$u_{tt} - q - s - 12uq - 12p^2 = 0, \quad (3.17)$$

$$v_{tt} - p - r - 12up = 0, \quad (3.18)$$

$$p = u_x, q = p_x, r = q_x, s = r_x. \quad (3.19)$$

From the above discussion, we have the following theorem for the Boussinesq equation:

Theorem 3.1 *The system (3.9)-(3.10) is an integrable discretization of the Boussinesq equation (3.5). Using transformations $w_n = 2\ln(f_n)$, $v_n = w_{n,x}$, $u_n = v_{n,x}$, $p_n = u_{n,x}$, $q_n = p_{n,x}$, $r_n = q_{n,x}$, this system can be transformed to (3.11)-(3.16), which converge to the extended Boussinesq equation (3.1)-(3.3) as $h \rightarrow 0$.*

For the system (3.9)-(3.10), we have the following proposition:

Proposition 3 *The bilinear equations (3.9)-(3.10) have the Bäcklund transformation*

$$(D_t - aD_x^2)f_n \cdot g_n = \lambda f_n g_n, \quad (3.20)$$

$$(aD_x D_t - D_x^3 - (\lambda a + 1)D_x)f_n \cdot g_n = 0, \quad (3.21)$$

$$(D_x e^{-\frac{h}{2}D_n})f_n \cdot g_n = \left(-\frac{1}{h}e^{-\frac{h}{2}D_n} + \beta e^{\frac{h}{2}D_n}\right)f_n \cdot g_n, \quad (3.22)$$

where β and λ are arbitrary constants.

Proof. Let f_n be a solution of (3.9)-(3.10) and g_n be given by (3.20)-(3.22). Just like the proof of **Proposition 1**, we can have

$$\begin{aligned} P_1 &\equiv (D_t^2 - D_x^2 - D_x^4)g_n \cdot g_n = 0, \\ P_2 &\equiv (D_t - aD_x^2 + \frac{2a}{h}D_x)e^{\frac{h}{2}D_n}g_n \cdot g_n = 0, \end{aligned}$$

which means that equations (3.20)-(3.22) construct a Bäcklund transformation of (3.9)-(3.10).

We can derive a Lax pair for the nonlinear equations (3.11)-(3.16) from the Bäcklund transformation (3.20)-(3.22). Setting $v_n = (\ln g_n)_x$, $u_n = v_{n,x}$, $p_n = u_{n,x}$, $q_n = p_{n,x}$, $f_n = \phi_n g_n$, $\psi_n = \phi_{n,x}$ and $\Delta_n = \psi_{n,x}$ in (3.20)-(3.22), we get

$$\beta \begin{pmatrix} \phi_{n+1} \\ \psi_{n+1} \\ \Delta_{n+1} \end{pmatrix} = L_n \begin{pmatrix} \phi_n \\ \psi_n \\ \Delta_n \end{pmatrix}$$

and

$$\begin{pmatrix} \phi_{n,t} \\ \psi_{n,t} \\ \Delta_{n,t} \end{pmatrix} = Q_n \begin{pmatrix} \phi_n \\ \psi_n \\ \Delta_n \end{pmatrix},$$

where

$$\begin{aligned} L_n &= \begin{pmatrix} \frac{1}{h} + v_n - v_{n+1} & 1 & 0 \\ u_n - u_{n+1} & \frac{1}{h} + v_n - v_{n+1} & 1 \\ -\frac{1}{2}p_n - p_{n+1} + \frac{a}{2}v_{n,t} & -\frac{1}{4} - u_n - 2u_{n+1} & \frac{1}{h} + v_n - v_{n+1} \end{pmatrix} \\ Q_n &= \begin{pmatrix} \lambda + 2au_n & 0 & a \\ \frac{a}{2}p_n - \frac{3}{2}v_{n,t} & \lambda - au_n - \frac{a}{4} & 0 \\ \frac{a}{2}q_n - \frac{3}{2}u_{n,t} & -\frac{a}{2}p_n - \frac{3}{2}v_{n,t} & \lambda - au_n - \frac{a}{4} \end{pmatrix}. \end{aligned}$$

The compatibility condition of the two linear problems above is $L_{n,t} = Q_{n+1}L_n - L_nQ_n$, which produces the discrete equations (3.11)-(3.16).

4 Sawada-Kotera Equation

In this section we apply the new method to the extended Sawada-Kotera (SK) equation

$$v_t + s + 30uq + 60u^3 = 0, \quad (4.1)$$

$$u_t + \eta + 30pq + 30ur + 180u^2p = 0, \quad (4.2)$$

$$u = v_x, p = u_x, q = p_x, r = q_x, s = r_x. \quad (4.3)$$

The dependent variable transformation

$$u = (\ln f)_{xx}, v = (\ln f)_x, \quad (4.4)$$

gives the bilinear form

$$D_x(D_t + D_x^5)f \cdot f = 0. \quad (4.5)$$

A bilinear Bäcklund transformation of the extended SK equation is

$$(D_x^3 - \sigma D_x^2 + \frac{1}{3}\sigma^2 D_x + \lambda)f \cdot g = 0, \quad (4.6)$$

$$(2D_t - 3D_x^5 + 5\sigma D_x^4 - \frac{5}{3}\sigma^2 D_x^3 + 15\lambda D_x^2 - 10\lambda\sigma D_x + \mu)f \cdot g = 0, \quad (4.7)$$

where κ, λ, σ and μ are arbitrary constants (see details in [19]). Setting $\lambda = 0$, $\mu = 0$, $\sigma = -3\kappa$, $f \rightarrow f_n$ and $g \rightarrow f_{n-h}$ in (4.5), (4.6) and (4.7), we arrive at the compatible system

$$D_x(D_t + D_x^5)f_n \cdot f_n = 0, \quad (4.8)$$

$$(D_x^3 + 3\kappa D_x^2 + 3\kappa^2 D_x)f_n \cdot f_{n-h} = 0, \quad (4.9)$$

$$(2D_t - 3D_x^5 - 15\kappa D_x^4 - 15\kappa^2 D_x^3)f_n \cdot f_{n-h} = 0. \quad (4.10)$$

Now rewrite (4.9) as

$$[D_x^3 \sinh(\frac{h}{2}D_n) + 3\kappa D_x^2 \cosh(\frac{h}{2}D_n) + 3\kappa^2 D_x \sinh(\frac{h}{2}D_n)]f_n \cdot f_n = 0.$$

Expanding this equation in powers of h , we obtain

$$[3\kappa D_x^2 + \frac{3h}{2}\kappa^2 D_x D_n + O(h)]f_n \cdot f_n = 0. \quad (4.11)$$

By imposing the convergence condition $D_x = D_n + O(h^k)$, $k \geq 1$, we find that $\kappa = -\frac{2}{h}$. Thus we get an integrable differential-difference system

$$D_x(D_t + D_x^5)f_n \cdot f_n = 0, \quad (4.12)$$

$$(D_x^3 - \frac{6}{h}D_x^2 + \frac{12}{h^2}D_x)f_n \cdot f_{n-h} = 0, \quad (4.13)$$

$$(2D_t - 3D_x^5 + \frac{30}{h}D_x^4 - \frac{60}{h^2}D_x^3)f_n \cdot f_{n-h} = 0. \quad (4.14)$$

Letting $v_n = (lnf_n)_x$, $u_n = v_{n,x}$, $p_n = u_{n,x}$, $q_n = p_{n,x}$, $r_n = q_{n,x}$, $s_n = r_{n,x}$, $\eta_n = s_{n,x}$, equations (4.12)-(4.13) can be transformed to

$$v_{n,t} + s_n + 30u_n q_n + 60u_n^3 = 0, \quad (4.15)$$

$$u_{n,t} + \eta_n + 30p_n q_n + 30u_n r_n + 180u_n^2 p_n = 0, \quad (4.16)$$

$$(p_{n+h} - p_n) + 3(v_{n+h} - v_n)(u_{n+h} + u_n) + (v_{n+h} - v_n)^3 - \frac{6}{h}((u_{n+h} + u_n) + (v_{n+h} - v_n)^2) + \frac{12}{h^2}(v_{n+h} - v_n) = 0, \quad (4.17)$$

$$(q_{n+h} - q_n) + 3(u_{n+h} - u_n)(u_{n+h} + u_n) + 3(v_{n+h} - v_n)(p_{n+h} + p_n) + 3(v_{n+h} - v_n)^2(u_{n+h} - u_n) - \frac{6}{h}((p_{n+h} + p_n) + 2(v_{n+h} - v_n)(u_{n+h} - u_n)) + \frac{12}{h^2}(u_{n+h} - u_n) = 0, \quad (4.18)$$

$$(r_{n+h} - r_n) + 3(p_{n+h} - p_n)(u_{n+h} + u_n) + 3(v_{n+h} - v_n)(q_{n+h} + q_n) + 6(u_{n+h} - u_n)(p_{n+h} + p_n) + 6(v_{n+h} - v_n)(u_{n+h} - u_n)^2 + 3(v_{n+h} - v_n)^2(p_{n+h} - p_n) - \frac{6}{h}((q_{n+h} + q_n) + 2(v_{n+h} - v_n)(p_{n+h} - p_n) + 2(u_{n+h} - u_n)^2) + \frac{12}{h^2}(p_{n+h} - p_n) = 0, \quad (4.19)$$

$$(s_{n+h} - s_n) + 3(q_{n+h} - q_n)(u_{n+h} + u_n) + 9(p_{n+h} - p_n)(p_{n+h} + p_n) + 9(u_{n+h} - u_n)(q_{n+h} + q_n) + 3(v_{n+h} - v_n)(r_{n+h} + r_n) + 6(u_{n+h} - u_n)^3 + 18(v_{n+h} - v_n)(u_{n+h} - u_n)(p_{n+h} - p_n) + 3(v_{n+h} - v_n)^2(q_{n+h} - q_n) - \frac{6}{h}((r_{n+h} + r_n) + 2(v_{n+h} - v_n)(q_{n+h} - q_n) + 6(u_{n+h} - u_n)(p_{n+h} - p_n)) + \frac{12}{h^2}(q_{n+h} - q_n) = 0, \quad (4.20)$$

$$(\eta_{n+h} - \eta_n) + 3(r_{n+h} - r_n)(u_{n+h} + u_n) + 12(q_{n+h} - q_n)(p_{n+h} + p_n) + 18(p_{n+h} - p_n)(q_{n+h} + q_n) + 12(u_{n+h} - u_n)(r_{n+h} + r_n) + 3(v_{n+h} - v_n)(s_{n+h} + s_n) + 36(u_{n+h} - u_n)^2(p_{n+h} - p_n) + 18(v_{n+h} - v_n)(p_{n+h} - p_n)^2 + 24(v_{n+h} - v_n)(u_{n+h} - u_n)(q_{n+h} - q_n) + 3(v_{n+h} - v_n)^2(r_{n+h} - r_n) - \frac{6}{h}((s_{n+h} + s_n) + 2(v_{n+h} - v_n)(r_{n+h} - r_n)) + 8(u_{n+h} - u_n)(q_{n+h} - q_n) + 6(p_{n+h} - p_n)^2 + \frac{12}{h^2}(r_{n+h} - r_n) = 0. \quad (4.21)$$

Replacing v_n by $v(x, t)$, v_{n+h} by $v(x + h, t)$, and similarly for u, p, q, r, s , as $h \rightarrow 0$, equations (4.15)-(4.21) become

$$v_t + s + 30uq + 60u^3 = 0, \quad (4.22)$$

$$u_t + \eta + 30pq + 30ur + 180u^2p = 0, \quad (4.23)$$

$$u = v_x, p = u_x, q = p_x, r = q_x, s = r_x. \quad (4.24)$$

Remark 6 For (4.17)-(4.21), both sides must be multiplied by h before taking the limit.

For the bilinear differential-difference equations (4.12)-(4.14), we have this proposition:

Proposition 4 The bilinear equations (4.12)-(4.14) have the Bäcklund transformation

$$D_x^3 f_n \cdot g_n = \lambda f_n g_n, \quad (4.25)$$

$$(2D_t - 3D_x^5 - 15\lambda D_x^2) f_n \cdot g_n = 0, \quad (4.26)$$

$$(D_x e^{\frac{h}{2}D_n} - D_x e^{-\frac{h}{2}D_n}) f_n \cdot g_n = \frac{2}{h} (e^{\frac{h}{2}D_n} + e^{-\frac{h}{2}D_n}) f_n \cdot g_n, \quad (4.27)$$

$$\begin{aligned} & (D_x^3 e^{\frac{h}{2}D_n} - D_x^3 e^{-\frac{h}{2}D_n} - \frac{6}{h} D_x^2 e^{\frac{h}{2}D_n} - \frac{6}{h} D_x^2 e^{-\frac{h}{2}D_n} \\ & + 2\lambda e^{\frac{h}{2}D_n} - 2\lambda e^{-\frac{h}{2}D_n}) f_n \cdot g_n = 0, \end{aligned} \quad (4.28)$$

where λ is an arbitrary constant.

Proof. Let f_n be a solution of (4.12)-(4.14) and g_n be given by (4.25)-(4.28). If we can show that

$$\begin{aligned} P_1 &\equiv D_x(D_t + D_x^5)g_n \cdot g_n = 0, \\ P_2 &\equiv (D_x^3 - \frac{6}{h}D_x^2 + \frac{12}{h^2}D_x)e^{\frac{h}{2}D_n}g_n \cdot g_n = 0, \\ P_3 &\equiv (2D_t - 3D_x^5 + \frac{30}{h}D_x^4 - \frac{60}{h^2}D_x^3)e^{\frac{h}{2}D_n}g_n \cdot g_n = 0, \end{aligned}$$

then equations (4.25)-(4.28) construct a Bäcklund transformation of (4.12)-(4.14). It is obvious that $P_1 = 0$ as (4.25)-(4.26) simply form a Bäcklund transformation of $D_x(D_t + D_x^5)f_n \cdot f_n = 0$ (see (4.6) and (4.7)).

By (4.25)-(4.28) and the bilinear identities (A.1), (A.2), (A.6), we can precisely demonstrate that

$$\begin{aligned} & -(e^{\frac{h}{2}D_n} f_n \cdot f_n) P_2 \\ & \equiv [(D_x^3 - \frac{6}{h}D_x^2 + \frac{12}{h^2}D_x)e^{\frac{h}{2}D_n} f_n \cdot f_n] (e^{\frac{h}{2}D_n} g_n \cdot g_n) \\ & \quad - (e^{\frac{h}{2}D_n} f_n \cdot f_n) [(D_x^3 - \frac{6}{h}D_x^2 + \frac{12}{h^2}D_x)e^{\frac{h}{2}D_n} g_n \cdot g_n] \\ & = -3D_x [(D_x e^{\frac{h}{2}D_n} f_n \cdot g_n - \frac{2}{h} e^{\frac{h}{2}D_n} f_n \cdot g_n) \cdot (D_x e^{-\frac{h}{2}D_n} f_n \cdot g_n + \frac{2}{h} e^{-\frac{h}{2}D_n} f_n \cdot g_n)] \\ & = 0. \end{aligned}$$

By using (4.13), (4.25)-(4.28) and the bilinear identities (A.1), (A.3), (A.6), (A.7), (A.11), (A.12)

$$\begin{aligned}
& -(e^{\frac{h}{2}D_n} f_n \cdot f_n) P_3 \\
& \equiv [(2D_t - 3D_x^5 + \frac{30}{h}D_x^4 - \frac{60}{h^2}D_x^3)e^{\frac{h}{2}D_n} f_n \cdot f_n](e^{\frac{h}{2}D_n} g_n \cdot g_n) \\
& \quad - (e^{\frac{h}{2}D_n} f_n \cdot f_n)[(2D_t - 3D_x^5 + \frac{30}{h}D_x^4 - \frac{60}{h^2}D_x^3)e^{\frac{h}{2}D_n} g_n \cdot g_n] \\
& = 15D_x[(D_x^3 e^{\frac{h}{2}D_n} - D_x^3 e^{-\frac{h}{2}D_n} - \frac{6}{h}D_x^2 e^{\frac{h}{2}D_n} - \frac{6}{h}D_x^2 e^{-\frac{h}{2}D_n} + 2\lambda e^{\frac{h}{2}D_n} \\
& \quad - 2\lambda e^{-\frac{h}{2}D_n})f_n \cdot g_n] \cdot (D_x e^{-\frac{h}{2}D_n} f_n \cdot g_n + \frac{2}{h}e^{-\frac{h}{2}D_n} f_n \cdot g_n) = 0.
\end{aligned}$$

Thus we have completed the proof. Here we omitted the detailed calculation.

From the Bäcklund transformation (4.25)-(4.26), we can derive a Lax pair for the system (4.15)-(4.21). Setting $v_n = (\ln g_n)_x$, $u_n = v_{n,x}$, $p_n = u_{n,x}$, $q_n = p_{n,x}$, $r_n = q_{n,x}$, $s_n = r_{n,x}$, $\eta_n = s_{n,x}$, $f_n = \phi_n g_n$, $\psi_n = \phi_{n,x}$, $\Delta_n = \psi_{n,x}$ in (4.25)-(4.28), we have

$$L_{1,n} \begin{pmatrix} \phi_{n+1} \\ \psi_{n+1} \\ \Delta_{n+1} \end{pmatrix} = L_{2,n} \begin{pmatrix} \phi_n \\ \psi_n \\ \Delta_n \end{pmatrix}$$

and

$$\begin{pmatrix} \phi_{n,t} \\ \psi_{n,t} \\ \Delta_{n,t} \end{pmatrix} = Q_n \begin{pmatrix} \phi_n \\ \psi_n \\ \Delta_n \end{pmatrix},$$

where

$$\begin{aligned}
L_{1,n} &= \begin{pmatrix} \frac{-2-hv_n+hv_{n+1}}{h} & \frac{1}{h} & 0 \\ \frac{-hu_n+hu_{n+1}}{h} & \frac{-2-hv_n+hv_{n+1}}{h} & 1 \\ a & b & \frac{2(2+hv_n-hv_{n+1})}{h} \end{pmatrix}, \\
L_{2,n} &= \begin{pmatrix} \frac{-2-hv_n+hv_{n+1}}{h} & \frac{1}{h} & 0 \\ \frac{-hu_n+hu_{n+1}}{h} & \frac{-2-hv_n+hv_{n+1}}{h} & 1 \\ c & d & -\frac{4+2hv_n-2hv_{n+1}}{h} \end{pmatrix}, \\
Q_n &= \begin{pmatrix} 36\lambda u_n & 6(q_n - 6u_n^2) & 9(\lambda - 2p_n) \\ 9\lambda(\lambda + 2p_n) & 6(r_n - 3(\lambda - 2p_n)u_n) & -12(q_n + 3u_n^2) \\ 6\lambda(q_n - 6u_n^2) & Q_n(3, 2) & -6(r_n + 3(\lambda + 2p_n)u_n) \end{pmatrix}
\end{aligned}$$

with

$$\begin{aligned}
Q_n(3, 2) &= 3(3\lambda^2 + 12p_n^2 + 2s_n + 36q_nu_n + 72u_n^3), \\
a &= \frac{-2h\lambda + 3u_n(2 + hv_n - hv_{n+1}) + 3u_{n+1}(2 + hv_n - hv_{n+1})}{h} \\
&\quad + \frac{(v_n - v_{n+1})^2(6 + hv_n - hv_{n+1})}{h}, \\
b &= \frac{-5hu_n - hu_{n+1} - 12v_n - 3hv_n^2 + 12v_{n+1} + 6hv_nv_{n+1} - 3hv_{n+1}^2}{h}, \\
c &= -\frac{2h\lambda + 3u_n(2 + hv_n - hv_{n+1}) + 3u_{n+1}(2 + hv_n - hv_{n+1})}{h} \\
&\quad - \frac{(v_n - v_{n+1})^2(6 + hv_n - hv_{n+1})}{h}, \\
d &= -\frac{hu_n + 5hu_{n+1} + 12v_n + 3hv_n^2 - 12v_{n+1} - 6hv_nv_{n+1} + 3hv_{n+1}^2}{h}.
\end{aligned}$$

The compatibility condition of the two linear problems above is

$$L_{2,n,t}L_{2,n}^{-1} - L_{1,n,t}L_{1,n}^{-1} = L_{1,n}Q_{n+1}L_{1,n}^{-1} - L_{2,n}Q_nL_{2,n}^{-1},$$

which produces the discrete equations (4.15)-(4.21).

As a conclusion of this section, we give the following theorem:

Theorem 4.1 *The system (4.12)-(4.14) is an integrable discretization of the extended SK equation (4.5). With $v_n = (\ln f_n)_x$, $u_n = v_{n,x}$, $p_n = u_{n,x}$, $q_n = p_{n,x}$, $r_n = q_{n,x}$, $s_n = r_{n,x}$, $\eta_n = s_{n,x}$, (4.12)-(4.14) can be transformed to (4.15)-(4.21), which converge to the extended SK equation (4.1)-(4.3) when we take a continuum limit.*

5 Ito Equation

In this section, we are going to discretize the extended Ito equation

$$w_{tt} + p_t + 6uv_t = 0, \quad (5.1)$$

$$v_t = w_{xt}, p_t = u_{xt}, u = v_x. \quad (5.2)$$

The dependent variable transformation

$$w = \ln f, v = (\ln f)_x, \quad (5.3)$$

gives the bilinear form

$$D_t(D_t + D_x^3)f \cdot f = 0, \quad (5.4)$$

which we also call it Ito equation without confusion. A bilinear Bäcklund transformation of (5.4) is [23] [20]

$$(D_x D_t - \mu D_x - \gamma D_t + \gamma \mu)f \cdot g = 0, \quad (5.5)$$

$$(D_t + 3\gamma^2 D_x - 3\gamma D_x^2 + D_x^3 - \lambda)f \cdot g = 0, \quad (5.6)$$

where λ , γ and μ are arbitrary constants. Setting $\mu = 0$, $\lambda = 0$, $\gamma = \frac{2}{h}$, $f \rightarrow f_n$ and $g \rightarrow f_{n-h}$ in (5.4), (5.5) and (5.6), we get an integrable differential-difference system

$$(D_t^2 + D_x^3 D_t) f_n \cdot f_n = 0, \quad (5.7)$$

$$(D_x D_t - \frac{2}{h} D_t) f_n \cdot f_{n-h} = 0, \quad (5.8)$$

$$(h D_t + h D_x^3 + \frac{12}{h} D_x - 6 D_x^2) f_n \cdot f_{n-h} = 0. \quad (5.9)$$

Considering x as an auxiliary variable, we next prove that when $h \rightarrow 0$, the nonlinear form of (5.7)-(5.9) converges to the extended Ito equation (5.1)-(5.2). With $w_n = \ln(f_n)$, $v_n = w_{n,x}$, $u_n = v_{n,x}$, $p_n = u_{n,x}$, $q_n = p_{n,x}$, equations (5.7)-(5.9) can be transformed to

$$w_{n,tt} + p_{n,t} + 6u_n v_{n,t} = 0, \quad (5.10)$$

$$(v_{n+h,t} + v_{n,t}) + (w_{n+h,t} - w_{n,t})(v_{n+h} - v_n) = \frac{2}{h}(w_{n+h,t} - w_{n,t}), \quad (5.11)$$

$$(p_{n+h,t} + p_{n,t}) + 3(u_{n+h,t} - u_{n,t})(v_{n+h} - v_n) + (w_{n+h,t} - w_{n,t})(p_{n+h} - p_n) = \frac{2}{h}(u_{n+h,t} - u_{n,t}), \quad (5.12)$$

$$6((u_{n+h} + u_n) + (v_{n+h} - v_n)^2) = \frac{12}{h}(v_{n+h} - v_n) + h(w_{n+h,t} - w_{n,t}) + h((p_{n+h} - p_n) + 3(v_{n+h} - v_n)(u_{n+h} + u_n) + (v_{n+h} - v_n)^3). \quad (5.13)$$

Replacing u_n by $u(x, t)$, u_{n+h} by $u(x+h, t)$, and similarly for w , v , p , as $h \rightarrow 0$, equations (5.10)-(5.13) become

$$w_{tt} + p_t + 6uv_t = 0, \quad (5.14)$$

$$v_t = w_{xt}, p_t = u_{xt}, u = v_x. \quad (5.15)$$

Thus we have

Theorem 5.1 *The system (5.7)-(5.9) form an integrable discretization of the Ito equation (5.4). With $w_n = \ln(f_n)$, $v_n = w_{n,x}$, $u_n = v_{n,x}$, $p_n = u_{n,x}$, $q_n = p_{n,x}$, (5.7)-(5.9) can be transformed to (5.10)-(5.13), which converge to the extended Ito equation (5.1)-(5.2) as $h \rightarrow 0$.*

For the bilinear differential-difference equations (5.7)-(5.9), we have the proposition

Proposition 5 *The bilinear equations (5.7)-(5.9) have the Bäcklund transformation*

$$(D_x e^{-\frac{h}{2} D_n} + \lambda D_x e^{\frac{h}{2} D_n} - \frac{2}{h} \lambda e^{\frac{h}{2} D_n} + \frac{2}{h} e^{-\frac{h}{2} D_n}) f_n \cdot g_n = 0, \quad (5.16)$$

$$(D_t e^{-\frac{h}{2} D_n} - \lambda D_t e^{\frac{h}{2} D_n} - \lambda \omega e^{\frac{h}{2} D_n} + \omega e^{-\frac{h}{2} D_n}) f_n \cdot g_n = 0, \quad (5.17)$$

$$(D_t + D_x^3) f_n \cdot g_n = 0, \quad (5.18)$$

$$(D_x D_t - \mu D_x) f_n \cdot g_n = 0, \quad (5.19)$$

where λ, ω, μ are arbitrary constants.

Proof. Let f_n be a solution of (5.7)-(5.9) and g_n be given by (5.16)-(5.19). If we can show that

$$\begin{aligned} P_1 &\equiv (D_t^2 + D_x^3 D_t) g_n \cdot g_n = 0, \\ P_2 &\equiv (D_x D_t - \frac{2}{h} D_t) e^{\frac{h}{2} D_n} g_n \cdot g_n = 0, \\ P_3 &\equiv (h D_t + h D_x^3 + \frac{12}{h} D_x - 6 D_x^2) e^{\frac{h}{2} D_n} g_n \cdot g_n = 0, \end{aligned}$$

then (5.16)-(5.19) form a Bäcklund transformation of (5.7)-(5.9). Noticing that equations (5.5)-(5.6) form a Bäcklund transformation of the Ito equation $(D_t^2 + D_x^3 D_t) f \cdot f = 0$, it is obvious that $P_1 = 0$. By using (5.16)-(5.17) and the bilinear identities (A.1), (A.8)-(A.10), we can calculate that

$$\begin{aligned} &-(e^{\frac{h}{2} D_n} f_n \cdot f_n) P_2 \\ &\equiv [(D_x D_t - \frac{2}{h} D_t) e^{\frac{h}{2} D_n} f_n \cdot f_n] (e^{\frac{h}{2} D_n} g_n \cdot g_n) \\ &\quad -(e^{\frac{h}{2} D_n} f_n \cdot f_n) [(D_x D_t - \frac{2}{h} D_t) e^{\frac{h}{2} D_n} g_n \cdot g_n] \\ &= \frac{1}{2} D_x [(D_t e^{-\frac{h}{2} D_n} f_n \cdot g_n - \lambda D_t e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (e^{\frac{h}{2} D_n} f_n \cdot g_n - \lambda^{-1} e^{-\frac{h}{2} D_n} f_n \cdot g_n)] \\ &= 0. \end{aligned}$$

By (5.16), (5.18) and the bilinear identities (A.1), (A.2), (A.6),

$$\begin{aligned} &-(e^{\frac{h}{2} D_n} f_n \cdot f_n) P_3 \\ &\equiv [(D_t + \frac{12}{h^2} D_x - \frac{6}{h} D_x^2 + D_x^3) e^{\frac{h}{2} D_n} f_n \cdot f_n] (e^{\frac{h}{2} D_n} g_n \cdot g_n) \\ &\quad -(e^{\frac{h}{2} D_n} f_n \cdot f_n) [(D_t + \frac{12}{h^2} D_x - \frac{6}{h} D_x^2 + D_x^3) e^{\frac{h}{2} D_n} g_n \cdot g_n] \\ &= -3 D_x [(D_x e^{\frac{h}{2} D_n} f_n \cdot g_n - \frac{2}{h} e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (D_x e^{-\frac{h}{2} D_n} f_n \cdot g_n + \frac{2}{h} e^{-\frac{h}{2} D_n} f_n \cdot g_n)] \\ &= 0. \end{aligned}$$

Thus we have completed the proof.

From the Bäcklund transformation (5.16)-(5.19), we can derive a Lax pair for the discrete integrable system (5.10)-(5.13). Setting $\mu = -\omega$, $v_n = (l n g_n)_x$, $u_n = v_{n,x}$, $p_n = u_{n,x}$, $f_n = \phi_n g_n$, $\psi_n = \phi_{n,x}$, $\Delta_n = \psi_{n,x}$ and $\Sigma_n = \Delta_{n,x}$ in (5.16)-(5.19), we obtain

$$L_{1,n} \begin{pmatrix} \phi_{n+1} \\ \psi_{n+1} \\ \Delta_{n+1} \\ \Sigma_{n+1} \end{pmatrix} = L_{2,n} \begin{pmatrix} \phi_n \\ \psi_n \\ \Delta_n \\ \Sigma_n \end{pmatrix}$$

and

$$\begin{pmatrix} \phi_{n,t} \\ \psi_{n,t} \\ \Delta_{n,t} \\ \Sigma_{n,t} \end{pmatrix} = Q_n \begin{pmatrix} \phi_n \\ \psi_n \\ \Delta_n \\ \Sigma_n \end{pmatrix},$$

where

$$L_{1,n} = \begin{pmatrix} -\frac{2\lambda}{h} - \lambda v_n + \lambda v_{n+1} & \lambda & 0 & 0 \\ -\lambda u_n + \lambda u_{n+1} & -\frac{2\lambda}{h} - \lambda v_n + \lambda v_{n+1} & \lambda & 0 \\ -\lambda p_n + \lambda p_{n+1} & -2\lambda u_n + 2\lambda u_{n+1} & -\frac{2\lambda}{h} - \lambda v_n + \lambda v_{n+1} & \lambda \\ -\lambda\omega + \lambda w_{n,t} - \lambda w_{n+1,t} & 6\lambda u_{n+1} & 0 & \lambda \end{pmatrix},$$

$$L_{2,n} = \begin{pmatrix} -\frac{2}{h} - v_n + v_{n+1} & -1 & 0 & 0 \\ -u_n + u_{n+1} & -\frac{2}{h} - v_n + v_{n+1} & -1 & 0 \\ -p_n + p_{n+1} & -2(u_n - u_{n+1}) & -\frac{2}{h} - v_n + v_{n+1} & -1 \\ -\omega - w_{n,t} + w_{n+1,t} & 6u_n & 0 & 1 \end{pmatrix},$$

$$Q_n = \begin{pmatrix} 0 & -6u_n & 0 & -1 \\ -2v_{n,t} & -\omega & 0 & 0 \\ -2u_{n,t} & -2v_{n,t} & -\omega & 0 \\ -2p_{n,t} & -4u_{n,t} & -2v_{n,t} & -\omega \end{pmatrix}.$$

The compatibility condition of the two linear problems above is

$$L_{2,n,t}L_{2,n}^{-1} - L_{1,n,t}L_{1,n}^{-1} = L_{1,n}Q_{n+1}L_{1,n}^{-1} - L_{2,n}Q_nL_{2,n}^{-1},$$

which gives the discrete equations (5.10)-(5.13).

6 Conclusion and Discussion

In this paper, based on the bilinear method, we have presented a systematic procedure towards finding integrable discretizations. The key to this method focuses on deriving compatible equations. Here we have chosen the well-known Bäcklund transformation. We must remark that generating difference equations from Bäcklund transformations has been studied for many years. Only with the convergence condition ($D_x = D_n + O(h^k)$, $k \geq 1$ for the x -direction discretization) concerned, the difference equations may converge to the original equation and thus approach the integrable discretization.

This procedure would seem to merit further attention. Our plans include conducting numerical studies, and trying to work out a suitable inverse scattering formalism. Full discretization of soliton equations and the promotion to other kinds of equations, such as the NLS equation or the Lotka-Volterra system will also be considered.

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Appendix A Bilinear Operator Identities

$$\begin{aligned}
& (D_x e^{\frac{\hbar}{2} D_n} f_n \cdot f_n)(e^{\frac{\hbar}{2} D_n} g_n \cdot g_n) - (e^{\frac{\hbar}{2} D_n} f_n \cdot f_n)(D_x e^{\frac{\hbar}{2} D_n} g_n \cdot g_n) \\
&= 2 \sinh\left(\frac{\hbar}{2} D_n\right)(D_x f_n \cdot g_n) \cdot (f_n g_n) \\
&= D_x(e^{\frac{\hbar}{2} D_n} f_n \cdot g_n) \cdot (e^{-\frac{\hbar}{2} D_n} f_n \cdot g_n). \tag{A.1}
\end{aligned}$$

$$\begin{aligned}
& (D_x^2 e^{\frac{\hbar}{2} D_n} f_n \cdot f_n)(e^{\frac{\hbar}{2} D_n} g_n \cdot g_n) - (e^{\frac{\hbar}{2} D_n} f_n \cdot f_n)(D_x^2 e^{\frac{\hbar}{2} D_n} g_n \cdot g_n) \\
&= D_x[(D_x e^{\frac{\hbar}{2} D_n} f_n \cdot g_n) \cdot (e^{-\frac{\hbar}{2} D_n} f_n \cdot g_n) \\
&\quad - (e^{\frac{\hbar}{2} D_n} f_n \cdot g_n) \cdot (D_x^{-\frac{\hbar}{2} D_n} f_n \cdot g_n)]. \tag{A.2}
\end{aligned}$$

$$\begin{aligned}
& D_x[(D_x e^{\frac{\hbar}{2} D_n} f_n \cdot g_n) \cdot (e^{-\frac{\hbar}{2} D_n} f_n \cdot g_n) + (e^{\frac{\hbar}{2} D_n} f_n \cdot g_n) \cdot (D_x^{-\frac{\hbar}{2} D_n} f_n \cdot g_n)] \\
&= 2 \sinh\left(\frac{\hbar}{2} D_n\right)(D_x^2 f_n \cdot g_n) \cdot (f_n g_n). \tag{A.3}
\end{aligned}$$

$$\begin{aligned}
& (D_x D_y e^{\frac{\hbar}{2} D_n} f_n \cdot f_n)(e^{\frac{\hbar}{2} D_n} g_n \cdot g_n) - (e^{\frac{\hbar}{2} D_n} f_n \cdot f_n)(D_x D_y e^{\frac{\hbar}{2} D_n} g_n \cdot g_n) \\
&= D_y[(D_x e^{\frac{\hbar}{2} D_n} f_n \cdot g_n) \cdot (e^{-\frac{\hbar}{2} D_n} f_n \cdot g_n) - (e^{\frac{\hbar}{2} D_n} f_n \cdot g_n) \cdot (D_x e^{-\frac{\hbar}{2} D_n} f_n \cdot g_n)] \\
&\quad + (D_x e^{\frac{\hbar}{2} D_n} f_n \cdot f_n) \cdot (D_y e^{\frac{\hbar}{2} D_n} g_n \cdot g_n) \\
&\quad - (D_y e^{\frac{\hbar}{2} D_n} f_n \cdot f_n) \cdot (D_x e^{\frac{\hbar}{2} D_n} g_n \cdot g_n). \tag{A.4}
\end{aligned}$$

$$\begin{aligned}
& D_y[(D_x e^{\frac{\hbar}{2} D_n} f_n \cdot g_n) \cdot (e^{-\frac{\hbar}{2} D_n} f_n \cdot g_n) + (e^{\frac{\hbar}{2} D_n} f_n \cdot g_n) \cdot (D_x^{-\frac{\hbar}{2} D_n} f_n \cdot g_n)] \\
&= 2 \sinh\left(\frac{\hbar}{2} D_n\right)[(D_x D_y f_n \cdot g_n) \cdot (f_n g_n) + (D_y f_n \cdot g_n) \cdot (D_x f_n \cdot g_n)]. \tag{A.5}
\end{aligned}$$

$$\begin{aligned}
& (D_x^3 e^{\frac{\hbar}{2} D_n} f_n \cdot f_n)(e^{\frac{\hbar}{2} D_n} g_n \cdot g_n) - (e^{\frac{\hbar}{2} D_n} f_n \cdot f_n)(D_x^3 e^{\frac{\hbar}{2} D_n} g_n \cdot g_n) \\
&= 2 \sinh\left(\frac{\hbar}{2} D_n\right)(D_x^3 f_n \cdot g_n) \cdot (f_n g_n) \\
&\quad - 3 D_x(D_x e^{\frac{\hbar}{2} D_n} f_n \cdot g_n) \cdot (D_x e^{-\frac{\hbar}{2} D_n} f_n \cdot g_n). \tag{A.6}
\end{aligned}$$

$$\begin{aligned}
& (D_x e^{\frac{h}{2} D_n} f_n \cdot f_n)(D_x^2 e^{\frac{h}{2} D_n} g_n \cdot g_n) - (D_x^2 e^{\frac{h}{2} D_n} f_n \cdot f_n)(D_x e^{\frac{h}{2} D_n} g_n \cdot g_n) \\
&= 2 \sinh\left(\frac{h}{2} D_n\right)(D_x f_n \cdot g_n) \cdot (D_x^2 f_n \cdot g_n) \\
&\quad + D_x(D_x e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (D_x e^{-\frac{h}{2} D_n} f_n \cdot g_n) \\
&= -D_x[(D_x^2 e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (e^{-\frac{h}{2} D_n} f_n \cdot g_n) \\
&\quad + (e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (D_x^2 e^{-\frac{h}{2} D_n} f_n \cdot g_n) \\
&\quad + (D_x e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (D_x e^{-\frac{h}{2} D_n} f_n \cdot g_n)]. \tag{A.7}
\end{aligned}$$

$$\begin{aligned}
& (D_z D_t e^{\frac{h}{2} D_n} f_n \cdot f_n)(e^{\frac{h}{2} D_n} g_n \cdot g_n) - (e^{\frac{h}{2} D_n} f_n \cdot f_n)(D_z D_t e^{\frac{h}{2} D_n} g_n \cdot g_n) \\
&= \frac{1}{2} D_z[(D_t e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (e^{-\frac{h}{2} D_n} f_n \cdot g_n) \\
&\quad - (e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (D_t e^{-\frac{h}{2} D_n} f_n \cdot g_n)] \\
&\quad + \frac{1}{2} D_t[(D_z e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (e^{-\frac{h}{2} D_n} f_n \cdot g_n) \\
&\quad - (e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (D_z e^{-\frac{h}{2} D_n} f_n \cdot g_n)]. \tag{A.8}
\end{aligned}$$

$$\begin{aligned}
& D_t[(D_z e^{-\frac{h}{2} D_n} f_n \cdot g_n) \cdot (e^{-\frac{h}{2} D_n} f_n \cdot g_n)] \\
&= D_z[(D_t e^{-\frac{h}{2} D_n} f_n \cdot g_n) \cdot (e^{-\frac{h}{2} D_n} f_n \cdot g_n)]. \tag{A.9}
\end{aligned}$$

$$\begin{aligned}
& D_t[(e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (D_z e^{\frac{h}{2} D_n} f_n \cdot g_n)] \\
&= D_z[(e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (D_t e^{\frac{h}{2} D_n} f_n \cdot g_n)]. \tag{A.10}
\end{aligned}$$

$$\begin{aligned}
& (D_x^5 e^{\frac{h}{2} D_n} f_n \cdot f_n)(e^{\frac{h}{2} D_n} g_n \cdot g_n) - (e^{\frac{h}{2} D_n} f_n \cdot f_n)(D_x^5 e^{\frac{h}{2} D_n} g_n \cdot g_n) \\
&= 2 \sinh\left(\frac{h}{2} D_n\right)[(D_x^5 f_n \cdot g_n) \cdot (f_n g_n) + 5(D_x^3 f_n \cdot g_n) \cdot (D_x^2 f_n \cdot g_n)] \\
&\quad - 5D_x[(D_x^3 e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (D_x e^{-\frac{h}{2} D_n} f_n \cdot g_n) \\
&\quad + (D_x e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (D_x^3 e^{-\frac{h}{2} D_n} f_n \cdot g_n)] \\
&\quad - 5[(D_x^3 e^{\frac{h}{2} D_n} f_n \cdot f_n)(D_x^2 e^{\frac{h}{2} D_n} g_n \cdot g_n) \\
&\quad - (D_x^2 e^{\frac{h}{2} D_n} f_n \cdot f_n)(D_x^3 e^{\frac{h}{2} D_n} g_n \cdot g_n)]. \tag{A.11}
\end{aligned}$$

$$\begin{aligned}
& (D_x^4 e^{\frac{h}{2} D_n} f_n \cdot f_n)(e^{\frac{h}{2} D_n} g_n \cdot g_n) - (e^{\frac{h}{2} D_n} f_n \cdot f_n)(D_x^4 e^{\frac{h}{2} D_n} g_n \cdot g_n) \\
&= D_x[(D_x^3 e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (e^{-\frac{h}{2} D_n} f_n \cdot g_n) - (e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (D_x^3 e^{-\frac{h}{2} D_n} f_n \cdot g_n) \\
&\quad - 3(D_x^2 e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (D_x e^{-\frac{h}{2} D_n} f_n \cdot g_n) \\
&\quad + 3(D_x e^{\frac{h}{2} D_n} f_n \cdot g_n) \cdot (D_x^2 e^{-\frac{h}{2} D_n} f_n \cdot g_n)] \\
&\quad - 2[(D_x^3 e^{\frac{h}{2} D_n} f_n \cdot f_n)(D_x e^{\frac{h}{2} D_n} g_n \cdot g_n) \\
&\quad - (D_x e^{\frac{h}{2} D_n} f_n \cdot f_n)(D_x^3 e^{\frac{h}{2} D_n} g_n \cdot g_n)]. \tag{A.12}
\end{aligned}$$

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